Section 1.3 - Vertex, Degrees, and Counting

Friday, January 27, 2023 12:00 AM

Definition:

The *degree* of a vertex v in a graph G is written $d_G(v)$ or $d(v)$ and is the number of edges incident to v, with loops counted twice.

Definition:

Given a graph G, we write $\Delta(G)$ as the maximum degree of vertices and $\delta(G)$ the minimum degree

Definition:

G is *regular* if $\Delta(G) = \delta(g)$

G is k-regular if it is regular with $\Delta(G) = k$

Definition:

The *neighborhood* of a vertex v is $N_G(v)$ or $N(v)$ the set of vertices adjacent to v

It's also useful for us to clarify our notation for the size of a graph.

Definition:

A graph has *order* $n(G)$ if it has n vertices.

A graph has *size* $e(G)$ if it has e edges.

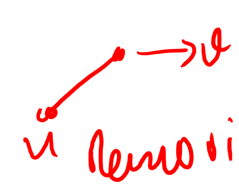
Notation:

We'll sometimes refer to the set $[n] = \{1,2,3, \dots, n\}$

Proposition:

If G is a graph, $\sum_{v \in G} d(v) = 2e(G)$

↳ degree sum formula



Removing each edge will remove 1 from the degree of a pair of vertices.

Corollary:

Every graph has an even number of vertices with odd degree. → clear

Corollary:

In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and in particular

$$\delta(G) \le \frac{2e(G)}{n(G)} \le \Delta(G)$$

Question:

How many edges does an order n k-regular graph have? → $\frac{nk}{2}$

Next: the hypercube is a k-regular graph.

Hypercubes: Called Q_k

$V(G) = \{0,1\}^k$ $\quad xy \in E(G)$ iff $\quad x-y = \pm e_i$

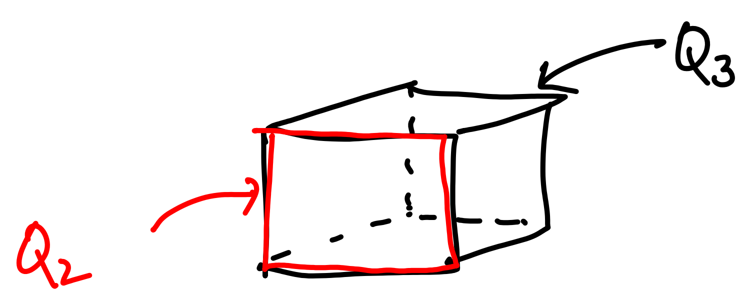
$e_1 = (1,0 \ldots)$ $e_2 = (0,1,\ldots)$

Is Q_k k-regular? You can flip any of the k coordinates to get a neighbor: YES.

$n(Q_k) = 2^k \Rightarrow 2e(G^k) = k\,2^k$

A subcube is a subgraph $\simeq Q_j$

isomorphic



Can keep any $k-j$ coordinates fixed and vary the others: (2^j many) to obtain a subcube $\Rightarrow \binom{k}{k-j} 2^{k-j}$ many subcubes.

The 2^{k-j} comes from the fact that you have to keep $k-j$ values fixed, but there are 2^{k-j} possibilities to put in those fixed coordinates.

for $j=1$ $\quad \binom{k}{k-j} 2^{k-j} = \binom{k}{k-1} 2^{k-1} = k\,2^{k-1}$

* Good HW problem.
Exam

* Is the hypercube bipartite?

Since Q_n is a bipartite graph (Exercise!), here is a comment on the structure of bipartite graphs.

Proposition:

If $k > 0$, any k-regular bipartite graph has the same number of vertices in any partite set.

Proof:

there are $k|X|$ many

All edges are between the partite sets. Count them based on edges in the first, then by edges in the second to establish equivalence.

there are $k|Y|$ many; $e(G) = k|X| = k|Y|$

worth skipping

Clever counting methods can be extremely helpful in analyzing graphs. (maybe exclude this example?)

Proposition:

The Petersen graph has 10 6-cycles.

Proof:

(Recall the Petersen graph)

Note that the Petersen graph manifestly contains 10 copies of the 'claw', one centered at each vertex

G has girth 5, so every 6-cycle is an induced subgraph - each point in such a cycle is adjacent to one external vertex. In Petersen graph, nonadjacent vertices have a unique common neighbor - take opposite points in the cycle.

Subtract cycle from G, 4 vertices remain, common neighbors have degree 1, last vertex has degree 3 - this is a claw.

To show each claw is obtained only once by this procedure, take a claw. Degree 1 vertices in it all have a common neighbor, so external vertices cannot coincide. Subtract claw from G, remaining graph is 2-regular, must be a 6-cycle.

These types of counting arguments have incredible power in graph theory.

Vertex deleted subgraphs and the Reconstruction Conjecture

(subgraphs of a graph with a single deleted vertex are sometimes called *vertex-deleted subgraphs*)

Proposition: (for them)
Let G be a simple graph with vertices $v_1, \ldots, v_n$ and $n \geq 3$. Then

$$e(G) = \frac{\sum_{v \in V(G)} e(G-v)}{n-2}$$

Exercise for someone in class?

Pf: $e(G-v) = e(G) - d(v)$

$$\sum_v e(G-v) = n\,e(G) - \sum_{v \in V} d(v) = n\,e(G) - 2\,e(G)$$

degree-sum

Note:

$$d_G(v_j) = \underbrace{\frac{\sum e(G-v_i)}{n-2}}_{e(G)} - e(G-v_j)$$

So given information about all vertex deleted subgraphs:

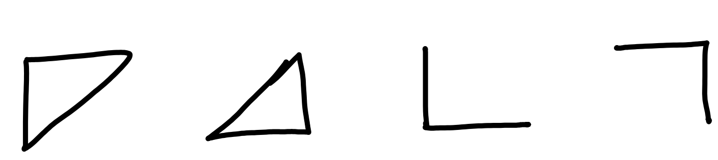
(all $e(G-v_j)$ for example) we can get

1) $e(G)$
2) $d_G(v_j)$ all vertex degrees.

It's reasonable to ask how well we can, given information about vertex-deleted subgraphs, figure out information about a graph itself.

Basically we are only given information like this

$\{C_3, C_3, P_3, P_3\}$



Conjecture: (Reconstruction Conjecture)
If G is a simple graph with at least 3 vertices, then G is uniquely determined by the list of (isomorphism classes of) its vertex deleted subgraphs.

(Due to Kelly & Ulam)
↳ super famous, incidentally the H Bomb is called the Teller-Ulam design

Extremal Problems (a type of counting I suppose)

We often wish to answer questions of the form: "How big (or small) of an example of ___ can be found among things of type ____?"
These are referred to as *extremal problems*

Proposition:

The minimum number of edges in a connected graph with n vertices is $n-1$

Proof:

We know if k edges at least $n-k$ components (from previous)

So $n-1$ edges is necessary for connectivity.

Is there a connected graph with exactly $n-1$ edges? Yes! The Path.

$$K = \min_{\substack{G:\, v(G)=n \\ G \text{ connected}}} e(G) = n-1$$

We showed $e(G) \geq n-1$ so $K \geq n-1$, and $\exists$ G st $e(G) = n-1$ satisfying the conditions.

Note:

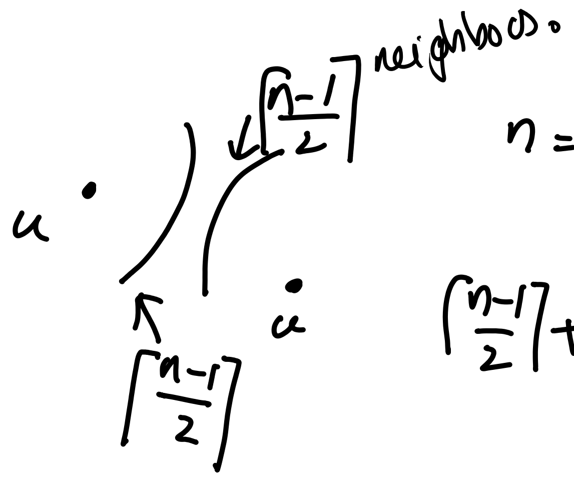
Somewhat formally written, to show that β is the minimum value of $f(G)$ over some class of graphs $\boldsymbol{G}$, then we must show two things.

- $f(G) \geq \beta$ for all $G \in \boldsymbol{G}$
- $f(G) = \beta$ for some $G \in \boldsymbol{G}$

Also solved by pigeonhole

Proposition:

If G is a simple order n graph with $\delta(G) \geq \frac{(n-1)}{2}$, then G is connected.



$n = 2k$

$2k+1$

$\left\lceil \frac{n-1}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil = \begin{cases} 2k & \text{even } n \\ 2k & \text{odd } n \end{cases}$

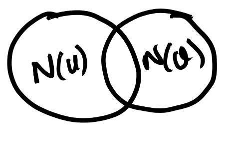
$\geq n-1$ in both cases.

If $v \in N(u)$, we're done, so assume $v \notin N(u)$.

$|N(u) \cup N(v)| = n-2$

$|N(u) \cap N(v)| \geq \left\lceil \frac{n-1}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil - (n-2)$

$\geq n-1-(n-2) = 1$



If $N(u)$ and $N(v)$ were disjoint, then $|N(u) \cup N(v)|$
$= |N(u)| + |N(v)| \geq n-1$

but this is 1 too many, so they must be disjoint.

Proof:

Any two vertices are adjacent or have a common neighbor

This proposition is actually part of an extremal problem in disguise.

Proposition:

The maximum value of $\delta(G)$ among simple disconnected graphs of order n is $\left\lfloor \frac{n}{2} \right\rfloor - 1$

Proof:

Let G be an n-vertex graph with two components, $K_{\lfloor \frac{n}{2} \rfloor}$ and $K_{\lceil \frac{n}{2} \rceil}$

Use the previous proposition for the rest.

Let $n = 2k+1$ $\quad K_k + K_{k+1} = G$

$\delta(G) = k-1 = \left\lfloor \frac{n}{2} \right\rfloor - 1$

disjoint union, see below.

Notice that in order to solve this extremal problem, we had to find an example for each value of n - a family of extremal solutions.

Definition:

The graph obtained by taking the union of graphs G and H with disjoint vertices is the *disjoint union* or *sum* $G + H$

The graph with m disjoint copies of G is mG

The previous proof used such a disjoint union.

Claim:

$K_n + K_m = \overline{K}_{m,n}$ ← complement of the biclique

Note:

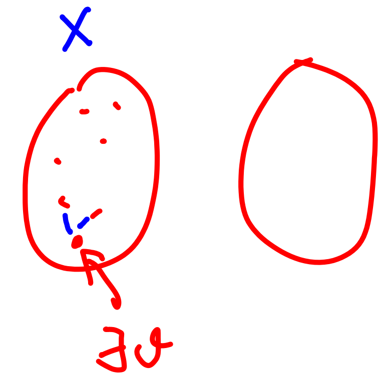
In a similar vein, we sometimes wish to find the largest example of some type within a single specified graph (largest clique, highest degree vertex, etc).

To distinguish these from extremal problems, we call them *optimization problems*

Solving such problems tends to be *very* complex. Proofs usually tend to describe an algorithm (sequence of steps) one could use to find such a optimum.

Theorem:

Every loopless graph G has a bipartite subgraph with at least $\frac{e(G)}{2}$ edges.



Proof:

Split $V(G)$ into two arbitrary sets X, Y. Take edges connecting these sets.
If this is not more than half the edges, there exists a vertex with more than half of adjacent vertices in the same set. Move it to the other set. Repeat. This process will terminate with a subgraph satisfying the desired property.

why must there exist such a vertex?

Suppose not $\sum_{v\in X}|N(v)\cap Y| \geq \sum_{v\in X}\frac{N(v)}{2}$

Similarly $\sum_{v\in Y}|N(v)\cap X| \geq \sum_{v\in Y}\frac{N(v)}{2}$

$\sum_{v\in X}\frac{N(v)}{2} + \sum_{v\in Y}\frac{N(v)}{2} = e(G)$ degree sum formula

2 · total # of edges between x and y

$\Rightarrow 2\cdot(\#\text{ of edges between } X \text{ and } Y) \geq e(G)$. So $\exists\, v \in X$ (wlog) st $|N(v)\cap X| > N(v)/2$

When moving from one set to the other

, you gain $|N(v)\cap X|$ edges and lose $|N(v)\cap Y|$.

Then $|N(v)\cap X| - |N(v)\cap Y|$
$= 2|N(v)\cap X| - |N(v)| > 0$ (net change in edges by this operation)

The next question could be motivated by a military question or by various social group questions. Imagine you have a collection of armies each with their own enemy armies. If no two distinct have a common enemy, how many "enemy connections" can there be?

Question:

How many edges can be there be in an order n graph which contains no triangles?

Definition:

A graph G is H-*free* if G has no induced subgraph isomorphic to H

Theorem:

The maximum number of edges in a C_3-free simple graph of order n is $\left\lfloor \frac{n^2}{4} \right\rfloor$

Proof:

Let G be such a graph, and x its vertex of maximal degree, k.
No two neighbors of x are adjacent, so each edge is between an element of $N(x)$ and its complement, or between two elements of its complement. Thus,

Every edge in the graph is accounted for in this sum, which includes x. Edges may be double counted

$$\sum_{v \notin N(x)} d(v) \geq e(G)$$

y z $\overline{N(x)}$ x z'

The former sum is at most $(n-k)k$ $\quad n - |N(x)|$

$e(G) \leq e(K_{n-k,k})$

The expression $(n-k)k$ is maximized when $k = \frac{n}{2}$. Does $\exists$ exist a graph with $\Delta(G) = k$, $e(G) = \frac{n^2}{4}$

what should k be to maximize $e(K_{n-k,k})$ (the edges in this biclique)

Let's try to prove this without calculus, though

The expression $(n-k)k$ represents the number of edges in $K_{n-k,k}$

Move a vertex from size k set to size $n-k$ set gains $k-1$ edges, loses $n-k$, so change f $2k - 1 - n$

$2k > n+1$ means change is positive

So for $2k - 1 - n \leq 0$,
You want $k \leq \frac{n+1}{2}$, so $k = \left\lceil \frac{n}{2} \right\rceil$

To achieve this maximum, consider

$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ $\quad e\left(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right) = \left\lfloor \frac{n^2}{4} \right\rfloor$

Note:

In principle, we could try to prove this kind of theorem by induction. However, it might be quite tricky and one has to be quite cautious. What goes wrong with the following argument for the previous theorem?

Base Case: $n = 2$ - this case is immediate
Induction Step:

Suppose the statement holds for $n = k$, so the complete graph $K_{\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor}$ is extremal

Even $k=2r$

$K_{r,r} \to K_{r+1,r}$

Odd $k=2r+1$

$K_{r,r+1} \to K_{r+1,r+1}$

in this case.
Add a new vertex to it to form a triangle-free graph with $k+1$ vertices
As long as new vertex is only adjacent to vertices from one partite set, this creates no triangles. This gives the new complete bipartite graph, completing the proof.

$K_{\lfloor (k+1)/2 \rfloor, \lceil (k+1)/2 \rceil}$

We are showing that if G contains $K_{\lceil \frac{k}{2} \rceil, \lfloor k/2 \rfloor}$ then adding a vertex shows the bound is not still satisfied.

Takeaway: If doing an inductive proof - make sure you start with an arbitrary thing satisfying the $k+1$-case and attempt to shrink it, rather than trying to grow the k-case.
Schematically, if the induction proof is for the claim $A(n) \Rightarrow B(n)$ for all G of size n

Skip.

$$G \text{ satisfies } A(n) \Rightarrow G' \text{ satisfies } A(n-1) \Rightarrow G' \text{ satisfies } B(n-1) \Rightarrow G \text{ satisfies } B(n)$$

Graphic Sequences

We've previously used degrees as a way to think about graph isomorphism. But degree of a single vertex isn't a meaningful graph invariant. How can we phrase degree in such a way as to make it a graph invariant concept?

Definition:

The *degree sequence* of a graph is the list of vertex degrees, usually written in nonincreasing order

$$d_1 \leq d_2 \leq \cdots \leq d_n$$

Question:

Can any arbitrary degree sequence be realized by a graph

2,2,3,4?

Proposition:

Iff $\sum d_i$ is even, then the given sequence is the degree sequence of some graph.

Proof:

(Necessity) Degree sum formula — the # of odd vertices must be even.

(Sufficiency) Connect pairs of odd vertices. Then draw a whole bunch of loops. since each loop adds 2 to the degree

$$\sum d_i = \underbrace{\sum_{i: d_i \text{ even}} d_i}_{\text{even}} + \sum_{i:\text{odd}} d_i$$

← Better have an even # of terms (Prove odd × odd = even)

This theorem is *much* harder, and not actually true, if we don't allow loops or multiple edges. (Consider 2,0,0)

?

Definition:

A *graphic sequence* is a list of nonnegative numbers that is the degree sequence for a simple graph. A simple graph with a given degree sequence "realizes" that sequence

There are a lot of possible ways to characterize this. Multiple such characterizations can be found in Sierksma-Hoogeveen (1991)

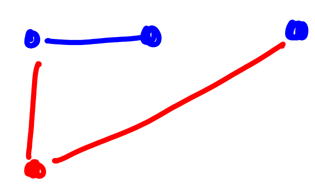
2211

101

reorder

110

Start with 4 vertices, take the vertex with maximal degree. If you remove the vertex and two of its edges, the new graph should have seq: 110



Idea:

How could we know if the sequence 3,3,3,3,3,2,2,1 is graphic?

Let's try to construct one - there's a vertex of degree 3 - could all of its neighbors also have degree 3?

If so, we could remove it from the graph, neighbors would now have degree 2. The resulting graph would have to have degree sequence

3,2,2,2,2,2,1

There's a vertex of degree 3, could all neighbors have degree 2?

If so, remove it, neighbors now have degree 1. Resulting graph would have degree sequence

2,2,1,1,1,1

Repeat

1,1,1,1,0

1,1,0,0

0,0,0

0,0

0

Draw each graph and build up

3 3 3 3 3 2 2 1

↓

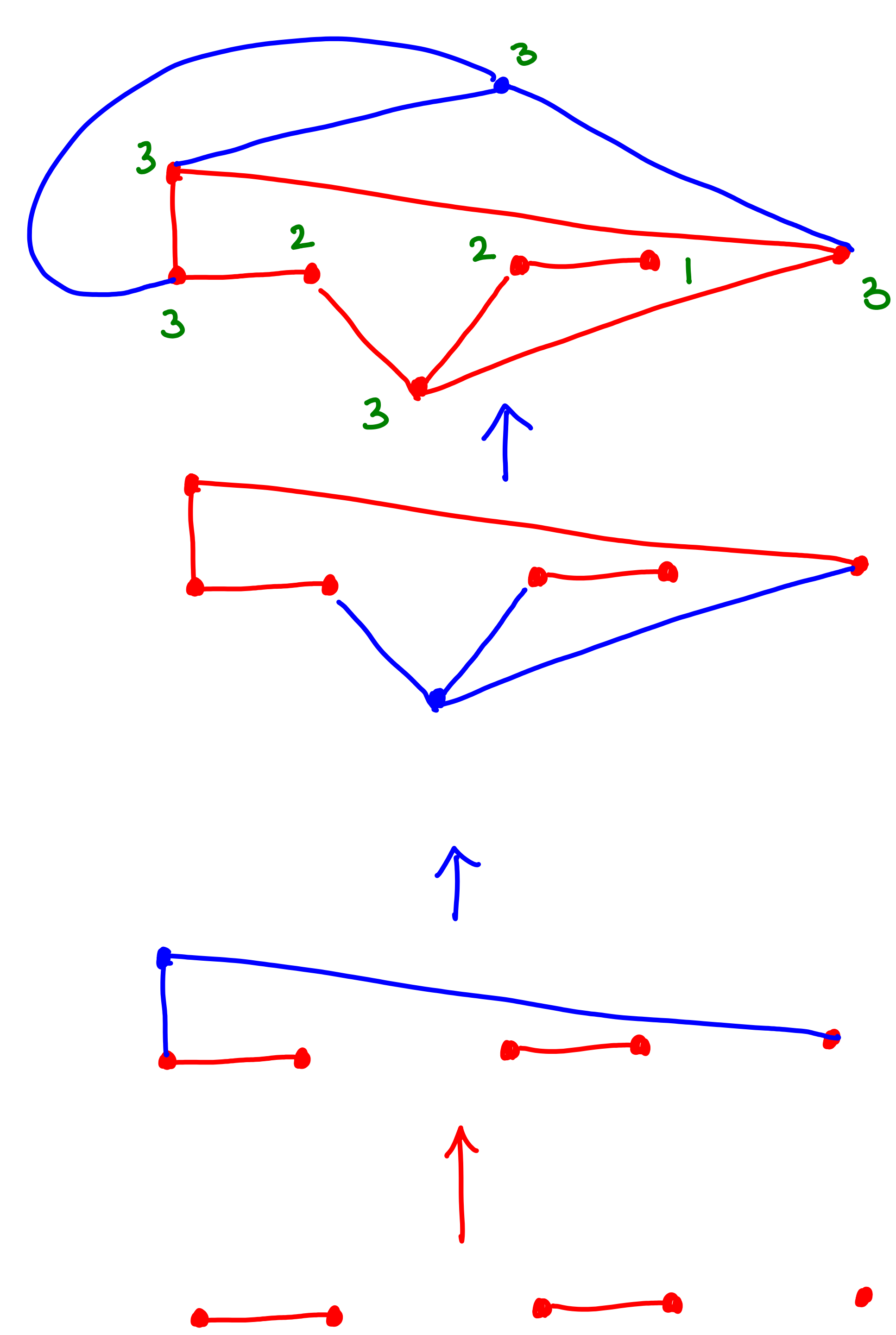
3 2 2 2 2 2 1

↓

2 2 1 1 1 1

↓

1 1 1 1 0



Theorem: (Havel [1955], Hakimi [1962])

For $n > 1$ an integer list d of size n is graphic if and only if d' is graphic, where d' is obtained from d by taking the largest element Δ of d and deleting it while decrementing the following Δ-largest elements by 1.

The only 1 element graphic sequence is $d_1 = 0$

Proof:

The 1 element case is trivial.

Suppose d' is realized by G', and d and d' have the above relationship.

Attach a new vertex to G' with Δ connections to vertices with degrees $d_2 - 1, d_3 - 1, \ldots, d_{\Delta+1} - 1$

Suppose d is realized by a graph G

Let $w \in V(G)$ have degree $d_1 = \Delta$

Let S be a set of vertices in G having degrees $d_2, d_3, \ldots, d_{\Delta+1}$

If $N(w) = S$, done

Else (modify G to increase $|N(w) \cap S|$)

There must exist $x \in S$ and $z \notin S$ with x not adjacent to w but z adjacent to w

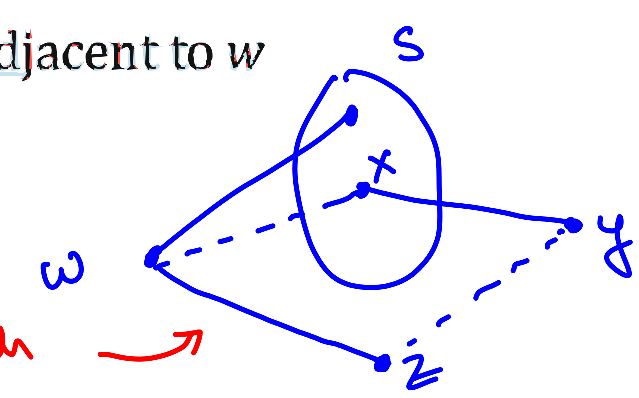
because it is in S and share degrees $d_2, \ldots, d_{\Delta+1}$ ← By necessity, $d(x) \geq d(z)$

$\Rightarrow$ $\exists$ vertex y connected to x but not z (since $z \sim w$)

Switch the connections up

Repeat previous step as necessary

2 switch →

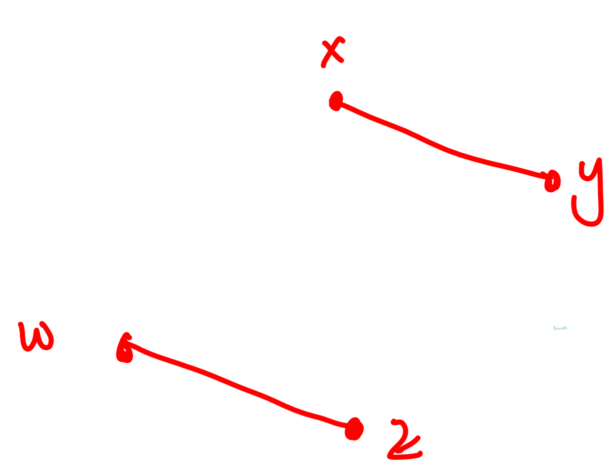


The procedure in the previous proof is kind of interesting, in that it codifies a particular way we can modify a graph without changing any vertex degrees.

Definition:

A *2-switch* is the replacement of a pair of edges xy and zw in a simple graph by edges yz and wx, given that neither of the latter were already in the graph

(draw)



Does not change vertex degrees.

These operations are surprisingly powerful!

same vertex set! So don't have to worry about relabeling.

Theorem: (Berge 1973)

If G, H are two simple graphs with vertex set V, then $d_G(v) = d_H(v)$ for every $v \in V$ if and only if there is a sequence of 2-switches that transforms G into H

Proof:

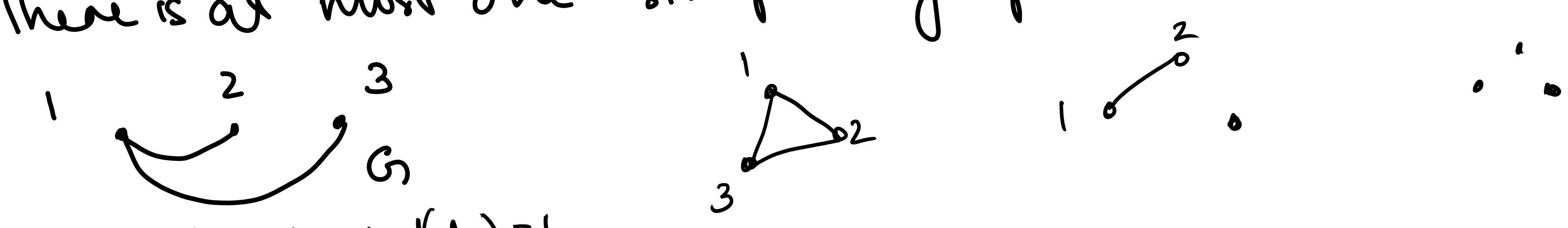
For sufficiency, 2-switches preserve vertex degree

For necessity, go by induction with $n = 3$ base case (degree sequence is a complete invariant here and below)

$d_1\ d_2\ d_3$

$\left.\begin{matrix} 2\,2\,2 \\ 2\,1\,1 \\ 1\,1\,0 \\ 0\,0\,0 \end{matrix}\right]$ List of degree sequences: allowable #s are 2,1,0. Enforce degree sum.

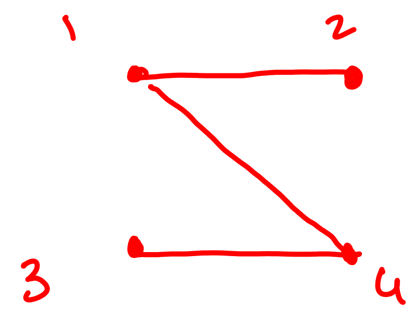
There is at most one simple graph with $d(v_i) = d_i$



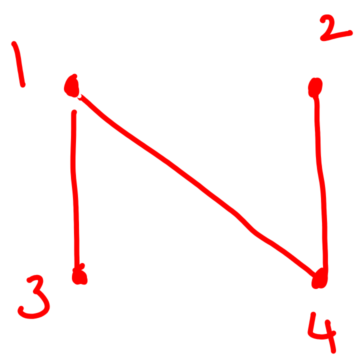
$d(v_1)=2 \quad d(v_2)=1 \quad d(v_3)=1$

Simplest nontrivial example.

n=4 case



2 1 1 1 $\rightarrow$



For $n \geq 4$

Transform both G and H using the previous strategy to ones where the highest degree vertex connects to the next highest degree vertices

Delete highest degree vertex from both, use induction on # of vertices